Journal of Engineering Physics and Thermophysics, Vol. 77, No. 4, 2004

INVESTIGATION OF THERMAL RESONANCE IN SOLUTION OF A TWO-DIMENSIONAL SINGULARLY PERTURBED BOUNDARY-VALUE PROBLEM OF NONSTATIONARY HEAT CONDUCTION WITH NONLINEAR BOUNDARY CONDITIONS OF THE STEFAN–BOLTZMANN TYPE

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Data of the analytical-numerical parametric investigation of a singularly perturbed temperature field in the boundary layer of the side of a rectangle on which nonlinear boundary conditions of the Stefan–Boltzmann type are specified have been given. It has been established that a nonuniform initial temperature distribution of the Gaussian type causes the appearance of "discontinuous traveling thermal waves" in the corresponding boundary layer. A set of parameters for which the "discontinuous traveling thermal waves," being superimposed, lead to a local nonlinear enhancement of the thermal field has been found. This effect can be considered as "thermal resonance."

Formulation of the Problem. In this work, we study the properties of a singularly perturbed (irregular) two-dimensional temperature field in the vicinity of a rectangle's boundary at which nonlinear boundary conditions of the Stefan–Boltzmann type are specified. In [1], we have proved a theorem establishing the Poincaré boundary-layer asymptotic expansion [2] of the solution of $T(x, y, t)$ when $\varepsilon \to 0$ with the following irregular boundary-value problem (in dimensionless variables):

$$\frac{\partial T}{\partial t} = \varepsilon \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right);$$

$$(x, y, t) \in \Omega_t = \left\{ (x, y, t) \colon a_{\rm r} < x < b\,,\ c < y < d\,,\ 0 < t < 1 \right\};$$

$$T(x, y, t) = T^0(x, y)\,, \quad t = +0\,, \quad (x, y) \in \overline{\Omega} = \left\{ (x, y) \colon a_{\rm r} \le x \le b\,, \quad c \le y \le d \right\};$$

$$\frac{\partial T(x, y, t)}{\partial x} = Q_1(t)\,, \quad x = a_{\rm r}\,, \quad c \le y \le d\,, \quad 0 < t < 1\,;$$

$$\frac{\partial T(x, y, t)}{\partial x} = Q_3(t)\,, \quad x = b\,, \quad c \le y \le d\,, \quad 0 < t < 1\,; \tag{1}$$

$$\frac{\partial T(x, y, t)}{\partial y} = Q_2(t)\,, \quad y = d\,, \quad a_{\rm r} \le x \le b, \quad 0 < t < 1\,;$$

$$\frac{\partial T(x, y, t)}{\partial y} = C_{\rm b}^{(c)} T^4(x, y, t)\,, \quad y = c\,, \quad a_{\rm r} \le x \le b\,, \quad 0 < t < 1\,.$$

Singularly perturbed nonlinear problems of the form (1) occur in solving a wide class of practical problems [3, 4]. The small parameter (perturbation) $\varepsilon = at_k/H^2$ appears in many cases [4], for example, in calculating thermal fields in structures in the case of fire [5, 6].

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 1062-0125/04/7704-0700

From Theorem 1 that was formulated and proved in [1], it follows that, when the natural assumptions (they are given below) hold, in the case $(x, y, t) \in$ "BNDLR (boundary layer)–c" (which means $y - c = O(\varepsilon^{1/2})$, $\varepsilon \to 0$, $\tilde{a} < x < \tilde{b}$, $a_r - \tilde{a} = O(\varepsilon)$, $b - \tilde{b} = O(\varepsilon)$, $\varepsilon \to 0$, and $\delta \le t < 1$, $\delta > 0$) the asymptotic expansion of Poincaré

$$T(x, y, t) \underset{\varepsilon\to 0}{\sim} \sum_{i=0}^{\infty} \overline{C}_i^{(c)}(x, y, t)\, \varepsilon^{i/2} + \exp\left\{-\frac{(y-c)^2}{4\varepsilon t}\right\} \sum_{i,j=0}^{\infty} \overline{C}_{ij}^{(c)}(x, y, t) \times$$

$$\times \varepsilon^{i/2}\left(\frac{y-c}{\sqrt{\varepsilon}}\right)^{j} + (y-c)\, R_0^{(c)}(x) + \varepsilon \sum_{i,j=1}^{\infty} \hat{C}_{ij1}^{(c)}(x)\, \varepsilon^{(2i-1)/2}\left(\frac{y-c}{\sqrt{\varepsilon}}\right)^{2j+1} +$$

$$+ \varepsilon \exp\left\{-\frac{(y-c)^2}{4\varepsilon t}\right\} \sum_{i,j=0}^{\infty} \check{C}_{ij1}^{(c)}(x, t)\, \varepsilon^{i/2}\left(\frac{y-c}{\sqrt{\varepsilon}}\right)^{2j+1} - \sqrt{\varepsilon}\, \exp\left\{-\frac{(y-c)^2}{4\varepsilon t}\right\} \sum_{i,j=0}^{\infty} \hat{C}_{ij2}^{(c)}(x, t)\, \varepsilon^{i}\left(\frac{y-c}{\sqrt{\varepsilon}}\right)^{2j} -$$

$$- \varepsilon \exp\left\{-\frac{(y-c)^2}{4\varepsilon t}\right\} \sum_{i,j=0}^{\infty} \check{C}_{ij2}^{(c)}(x, t)\, \varepsilon^{i/2}\left(\frac{y-c}{\sqrt{\varepsilon}}\right)^{2j}. \qquad (2)$$

holds true for the solution of $T(x, y, t)$ of boundary-value problem (1).

Expression (2) differs from formula (40) in [1] in the first two sums. This change has been made for the Poincaré asymptotics (2) to hold uniformly true for $t \in [0, 1]$. Its coefficients are not functions of the small parameter and are computed in explicit form. For example, we have

$$\overline{C}_0^{(c)}(x, y, t) = \frac{1}{2}\left[T^0(x, y) + T^0(x, 2c-y)\right], \quad \overline{C}_1^{(c)}(x, y, t) = \sqrt{\frac{t}{\pi}}\left[\frac{\partial T^0(x, y)}{\partial y} + \frac{\partial T^0(x, 2c-y)}{\partial y}\right],$$

$$\overline{C}_2^{(c)}(x, y, t) = \frac{t}{2}\left[\frac{\partial^2 T^0(x, y)}{\partial x^2} + \frac{\partial^2 T^0(x, 2c-y)}{\partial x^2} + \frac{\partial^2 T^0(x, y)}{\partial y^2} + \frac{\partial^2 T^0(x, 2c-y)}{\partial y^2}\right],$$

$$\overline{C}_{00}^{(c)}(x, y, t) = 0, \quad \overline{C}_{01}^{(c)}(x, y, t) = \frac{1}{2\sqrt{\pi t}}\left[T^0(x, y) - T^0(x, 2c-y)\right],$$

$$\overline{C}_{02}^{(c)}(x, y, t) = 0, \quad \overline{C}_{03}^{(c)}(x, y, t) = \frac{1}{3\cdot 2^2\sqrt{\pi t^3}}\left[T^0(x, y) - T^0(x, 2c-y)\right],$$

$$\overline{C}_{04}^{(c)}(x, y, t) = 0, \quad \overline{C}_{05}^{(c)}(x, y, t) = \frac{1}{5\cdot 3\cdot 2^3\sqrt{\pi t^5}}\left[T^0(x, y) - T^0(x, 2c-y)\right],$$

$$\overline{C}_{10}^{(c)}(x, y, t) = \frac{1}{2}\sqrt{\frac{t}{\pi}}\left[\frac{\partial T^0(x, y)}{\partial y} + \frac{\partial T^0(x, 2c-y)}{\partial y}\right],$$

$$\overline{C}_{20}^{(c)}(x, y, t) = 0, \quad \overline{C}_{21}^{(c)}(x, y, t) = \frac{\sqrt{t}}{2\sqrt{\pi}}\left[\frac{\partial^2 T^0(x, y)}{\partial x^2} - \frac{\partial^2 T^0(x, 2c-y)}{\partial x^2}\right],$$

$$\overline{C}_{22}^{(c)}(x, y, t) = 0, \quad \overline{C}_{23}^{(c)}(x, y, t) = \frac{1}{3\cdot 2^2\sqrt{\pi t}}\left[\frac{\partial^2 T^0(x, y)}{\partial x^2} - \frac{\partial^2 T^0(x, 2c-y)}{\partial x^2} + \frac{\partial^2 T^0(x, y)}{\partial y^2} - \frac{\partial^2 T^0(x, 2c-y)}{\partial y^2}\right],$$

$$\overline{C}_{24}^{(c)}(x, y, t) = 0\ ,\quad \overline{C}_{25}^{(c)}(x, y, t) = \frac{1}{5\cdot 3\cdot 2^3\sqrt{\pi t^3}}\left[\frac{\partial^2 T^0(x, y)}{\partial x^2} - \frac{\partial^2 T^0(x, 2c-y)}{\partial x^2} + \frac{\partial^2 T^0(x, y)}{\partial y^2} - \frac{\partial^2 T^0(x, 2c-y)}{\partial y^2}\right],$$

$$R_0^{(c)}(x) = C_{\mathrm{b}}^{(c)}\,[T^0(x, y)]^4\ . \tag{3}$$

The analytical expressions for the coefficients $\hat{C}_{ijk}^{(c)}$ and $\check{C}_{ijk}^{(c)}$, $k = 1, 2$, have been given in [1].

Parametric Analysis of the Boundary-Layer Values of the Irregular Thermal Field. This work primarily seeks to investigate the influence of nonlinear boundary conditions of the Stefan–Boltzmann type and initial conditions of the Gaussian type on the irregular temperature field of the rectangle. A similar investigation has been carried out in [7] for the case of nonlinear boundary conditions of the exponential (Arrhenius) type. It has turned out that nonlinear boundary conditions of the exponential (Arrhenius) type and a nonuniform temperature distribution lead to the appearance of two "discontinuous traveling thermal waves" in the corresponding boundary layer. It was established earlier in [8] that a nonlinear heat source of the exponential (Arrhenius) type analogously influences the solution of the corresponding irregular semilinear problem of Cauchy. In particular, in [8] we revealed a set of parameters for which two "discontinuous traveling thermal waves", being superimposed, lead to a local nonlinear enhancement ("thermal resonance").

Thus, we will assume that the initial distribution of $T^0(x, y)$ is prescribed by the analytical expression

$$T^0(x, y) = T(xH, yH) - T_0\ , \tag{4}$$

where

$$T_{\mathrm{in}}(x', y') = \left[\exp\left\{-\left(\frac{x'-l}{w_1}\right)^2\right\} + \exp\left\{-\left(\frac{x'+l}{w_1}\right)^2\right\}\right]\exp\left\{-\left(\frac{y'-y_0}{w_2}\right)^2\right\}; \tag{5}$$

here H, T_0, l, w_1, and w_2 are real numbers. Next, we assume that [1]

$$(x, y, t) \in \text{"BNDLR} - c\text{"}.$$

Furthermore, we assume that the function $T^0(x, y)$ has partial derivatives with respect to the arguments x and y of any order and is expanded in $\overline{\Omega}$ in a double Taylor series converging to the function by which it has been constructed. The functions $Q_k(t)$, $k = \overline{1, 3}$, are assumed to be continuous on [0, 1].

The aim of the investigation is as follows: by varying the parameters $C_{\mathrm{b}}^{(c)}$, l, y_0, and y, we must find such a combination of them that leads to a superposition and enhancement of the "discontinuous traveling thermal waves" in the "BNDLR–c," i.e., causes "thermal resonance." The parameters T_0, ε, w_1, w_2, a_{r}, b, c, and d are assumed to be constant.

The results of the analytical-numerical parametric investigation of the mutual influence of two peaks of the initial Gaussian distribution (4) on the temperature $T(x, y, t)$ in the "BNDLR–c" are presented in Figs. 1–3. The plots correspond to the following fixed values of the parameters: ε = 0.001, w_1 = 0.02, w_2 = 0.10, T_0 = 1, c = –1, and H = 1.00. The values of the dimensionless time t, selected in each series of figures, were as follows: t = 0, 0.1, 0.3, 0.6, and 0.9. Furthermore, the following values of the parameter l were fixed: l = 0.1, 0.04, 0.032, and 0.024.

An analysis of the plots of Fig. 1 shows that they are in qualitative agreement with the plots presented in the figures in [8], namely: in Fig. 1a (l = 0.1), each peak of the initial Gaussian temperature distribution is transformed into two "discontinuous traveling thermal waves" that do not interact by virtue of the relatively large initial distance (l = 0.1) between the initial peaks. This situation is analogous to that shown in the figure in [8] for a = 0.05. In Fig. 1b (l = 0.04), two "discontinuous traveling thermal waves," moving in opposition, are already superimposed, but, by virtue of the insufficiently small distance (l = 0.04), we do not observe the effect of maximum increase in the temperature $T(x, y, t)$ in Fig. 1b. This situation is analogous to that shown in the figures in [8] for a = 0.03 and for a = 0.02. In Fig. 1c (l = 0.032), two "discontinuous traveling thermal waves" moving in opposition, interact in such a manner in superposition that finally the temperature has its maximum ($T \approx 2$) at t = 0.9, whereas two "discontinuous traveling thermal waves" moving in opposite directions, have a maximum of $T(x, y, t) \approx 1$ (at t = 0.9), just as for l = 0.1. This situation is analogous to that shown in the figure in [8] for a = 0.016. The effect of enhancement of the

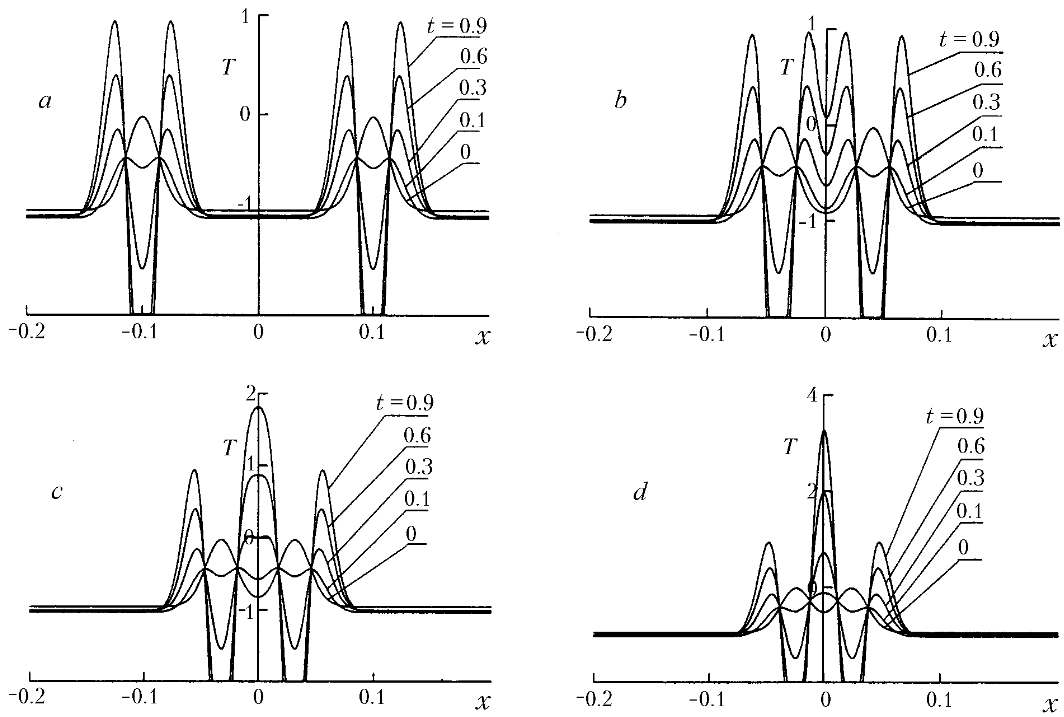

Fig. 1. Temperature T vs. dimensionless coordinate x for $\varepsilon = 0.001$, $w_1 = 0.02$, $w_2 = 0.1$, $y_0 = -1.000$, $y = -0.985$, $C_b^{(c)} = 3$, $T_0 = 1$, and different values of the dimensionless time t: a) $l = 0.1$; b) $l = 0.04$; c) $l = 0.032$; d) $l = 0.024$.

"discontinuous traveling thermal waves" is the greatest for $l = 0.024$ (see Fig. 1d). This situation is analogous to that shown in the figure in [8] for $a = 0.012$. The resulting maximum attains $T \approx 3$, i.e., is nearly three times as large as the maximum value of the "discontinuous traveling thermal waves" that have not interacted. Thus, from the plots given in Fig. 1 it follows that for $y_0 = -1.000$ nonlinear boundary conditions of the Stefan–Boltzmann type have the same "thermal-resonance" effect in the corresponding boundary layer as a nonlinear heat source of the exponential (Arrhenius) type does on solution of the Cauchy problem [8]. In [7], it has been shown that the "discontinuous traveling thermal waves" initiated by nonlinear boundary conditions of the exponential (Arrhenius) type and the initial Gaussian distribution occur within the corresponding boundary layer. As follows from Fig. 1, this phenomenon is also noted with nonlinear boundary conditions of the Stefan–Boltzmann type. It is clear from Fig. 2 that a growth in the parameter y_0 — the center of the initial Gaussian distribution — does not change the qualitative pattern of the "discontinuous traveling thermal waves" but their maximum values decrease. In Fig. 2 ($y_0 = -0.900$), the maximum is $T \approx 0.9$, i.e., is three times smaller than that in Fig. 1. Thus, we arrive at the substantiated conclusion: if the center y_0 of an initial temperature distribution of the Gaussian type is either at the rectangle's boundary at which nonlinear boundary conditions of the Stefan–Boltzmann type are specified or in the boundary layer of the rectangle, we have the effect of "discontinuous traveling thermal waves" inside the boundary layer and "thermal resonance" initiated by nonlinear boundary conditions of the Stefan–Boltzmann type and the nonuniform initial temperature distribution (see Fig. 1). If the center y_0 of an initial temperature distribution of the Gaussian type is beyond the corresponding boundary layer, the "thermal-resonance" effect is virtually absent (see Fig. 2).

Figure 3 shows the plots corresponding to $C_b^{(c)} = 0.1$ and $y_0 = -0.970$; the values of the remaining parameters are the same as those in Fig. 1. It is clear from these plots that the "thermal-resonance" effect in the "BNDLR–c" also occurs with substantial change in the constant $C_b^{(c)}$ appearing in nonlinear boundary conditions of the Stefan–Boltzmann type; the maximum value of the temperature remains virtually constant as compared to the case $C_b^{(c)} = 3$ (see Fig. 1).

The authors have carried out calculations and constructed plots for the following values of the parameters y_0 and y: $y_0 = -0.950$ and $y = 0.985$; $y_0 = -1.000$ and $y = 1.000$; $y_0 = -0.985$ and $y = 1.000$; the values of the remain-

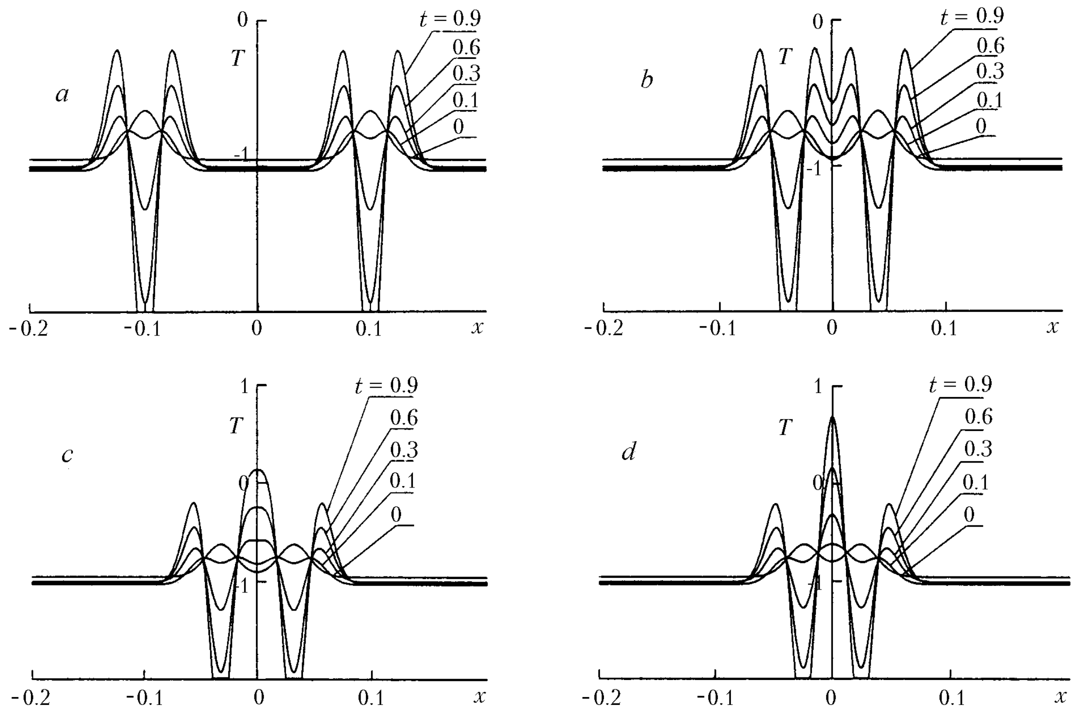

Fig. 2. Temperature T vs. dimensionless coordinate x for $\varepsilon = 0.001$, $w_1 = 0.02$, $w_2 = 0.1$, $y_0 = -0.900$, $y = -0.985$, $C_b^{(c)} = 3$, $T_0 = 1$, and different values of the dimensionless time t: a) $l = 0.1$; b) $l = 0.04$; c) $l = 0.032$; d) $l = 0.024$.

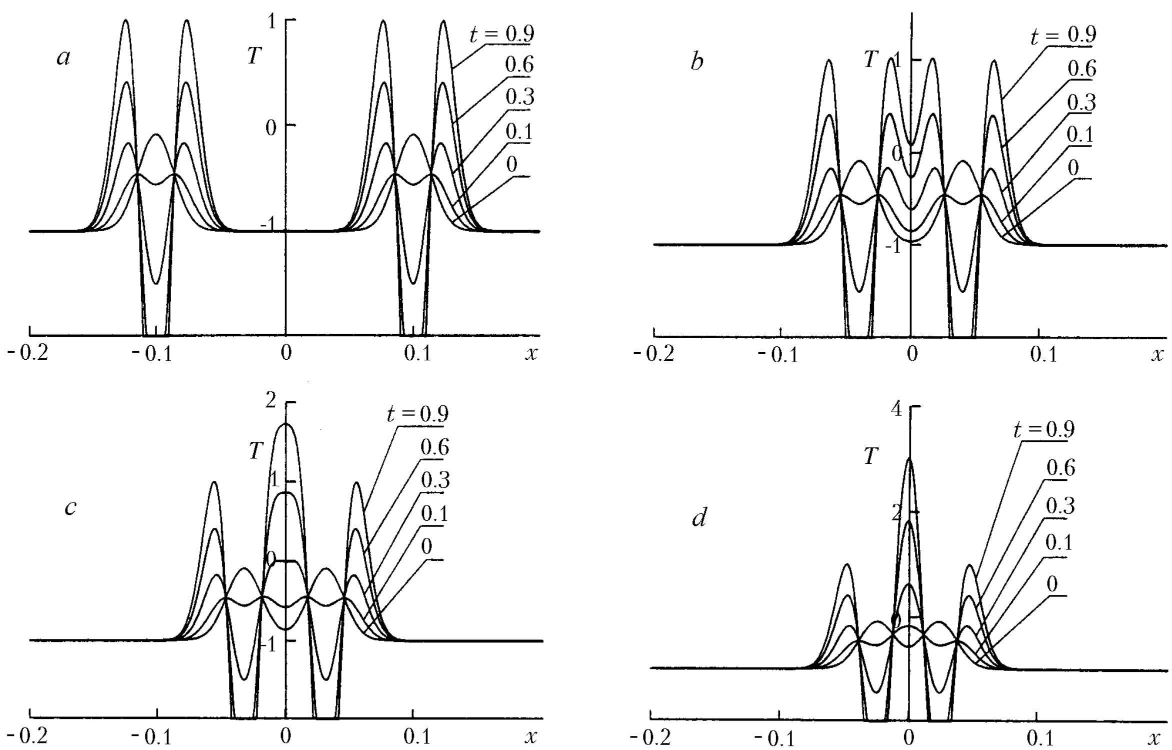

Fig. 3. Temperature T vs. dimensionless coordinate x for $\varepsilon = 0.001$, $w_1 = 0.02$, $w_2 = 0.1$, $y_0 = -1.000$, $y = -0.970$, $C_b^{(c)} = 0.1$, $T_0 = 1$, and different values of the dimensionless time t: a) $l = 0.1$; b) $l = 0.04$; c) $l = 0.032$; d) $l = 0.024$.

ing parameters are the same as those in the case of Fig. 1. The plots corresponding to these calculations have been omitted to save room; they confirm the correctness of the conclusion drawn above.

Discussion of the Results Obtained. The effect of "traveling waves" in the general case has long been known in the theory of semilinear equations of the parabolic type [9]. It has been described in [10, 11] for nonlinear heat sources of the exponential (Arrhenius) type. In [11], the regime of "traveling waves" has been called "self-oscillating propagation of a combustion wave in a condensed medium"; its physical interpretation in terms of combustion theory has been given in the same work. A self-similar solution in the form of a thermal wave has been given in [12] for the equation $U_t = \Delta U - U \ln U$. In [13], which is related to [12], it is noted that the process of combustion in the form of dissipative structures is accompanied by the phenomenon of self-focusing of combustion waves and by a decrease in the half-width of structures with time instead of the usual "spreading" of combustion in the medium; to excite combustion in the aggravation regime it is required that the size of the perturbation region for the given temperature maximum be no smaller than the "resonant length" L_T^*, i.e., the excitation of such combustion is "resonant" in character. Thus, we can assume that the superposition and enhancement of "discontinuous traveling thermal waves," which have been studied in [7, 8], are in agreement with the results of [9–13].

According to [3–6], the "thermal-resonance" effect established in this work has not been established earlier in investigations describing heat conduction with nonlinear boundary conditions of the Stefan–Boltzmann type. Instead of the term "thermal resonance," one can use the term "thermal self-focusing," proposed in [13].

It is safe to assume that the effect of "thermal resonance" ("thermal self-focusing") initiated by nonlinear boundary conditions of the Stefan–Boltzmann type occurs for more general models of heat and mass transfer. For example, one should expect "thermal self-focusing" in irregular fields in arbitrarily shaped bodies at the boundary of which nonlinear boundary conditions of both the exponential (Arrhenius) and Stefan–Boltzmann type are specified. The structure of the boundary-layer Poincaré asymptotics of such irregular thermal fields is established with the use of the results presented in [14–17]. An analogous assumption holds true for irregular heat-conduction problems with nonlinear boundary conditions on a moving boundary [18].

We note that the results obtained by other authors in investigating solutions of nonlinear problems of the type of (1) have been reviewed in [1]; the asymptotic expansions of Érdelyi (in terms of Érdelyi) of solutions of singularly perturbed partial equations are contained in [19] and differ from the asymptotic expansions of Poincaré (in terms of Poincaré) in that the coefficients of the asymptotic expansions of Érdelyi depend on the small parameter [2]. That is the reason why the Érdelyi asymptotic expansions give no way of making an analytical parametric analysis of the properties of solutions of the corresponding singularly perturbed mathematical models [19].

In [20], there are examples of the use of Poincaré asymptotics for analytical parametric analysis of the linear processes of heat and mass transfer.

In closing, the authors express their thanks to Professor É. M. Kartashov for stimulating discussions.

NOTATION

$T = T(x, y, t)$, dimensionless temperature of the rectangle; x, y, and t, dimensionless variables; a_r, b, c, and d, numbers prescribing the sides of the rectangle; $C_b^{(c)}$, constant; a, thermal diffusivity; t_k and H, time and space scales respectively; T_0, temperature scale; $T^0(x, y)$, function prescribing the initial conditions; $T^0(x, y) = T_{in}(x, y) - T_0$, $T_{in}(x, y) = \left[\exp\left\{-\left(\frac{x-l}{w_1}\right)^2\right\}\right] + \exp\left\{-\frac{(x+l)}{w_1}\right\}^2 \exp\left\{-\left(\frac{(y-y_0)}{w_2}\right)^2\right\}$; l, y_0, w_1, w_2, and T_0, parameters with the use of which the initial temperature distribution is prescribed; $\overline{C}_i^{(c)}(x, y, t)$, $\overline{C}_{ij}^{(c)}(x, y, t)$, $R_0^{(c)}(x)$, $\hat{C}_{ij1}^{(c)}(x)$, $\check{C}_{ij1}^{(c)}(x, t)$, $\hat{C}_{ij2}^{(c)}(x, t)$, and $\check{C}_{ijk}^{(c)}(x, t)$, coefficients of the asymptotic expansion; $Q_k(t)$, $k = \overline{1, 3}$, functions prescribing heat fluxes on the sides of the rectangle; U, solution of the semilinear equation $U_t = \Delta U - U \ln U$; L_T^*, resonant length; ε, dimensionless small parameter; Ω_t, space-time region; $\Omega_t = \{(x, y, t) : a_r < x < b, c < y < d, 0 < t < 1\}$; $\overline{\Omega}$, closure of the space region: $\overline{\Omega} = \{(x, y) : a_r \le x \le b, c \le y \le d\}$. Subscripts and superscripts: c points to the belonging to the boundary layer of the rectangle's side prescribed by the equation $y = c$; b points to the belonging to the boundary layer; 0, corresponds to the initial instant of time ($t = 0$); in, initial conditions; r, rectangle.

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